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# Supersymmetric dynamical invariants

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**Abstract**

We address the problem of identifying the (nonstationary) quantum systems that admit supersymmetric dynamical invariants. In particular, we give a general expression for the bosonic and fermionic partner Hamiltonians. Due to the supersymmetric nature of the dynamical invariant the solutions of the time-dependent Schrödinger equation for the partner Hamiltonians can be easily mapped to one another. We use this observation to obtain a class of exactly solvable time-dependent Schrödinger equations. As applications of our method, we construct classes of exactly solvable time-dependent generalized harmonic oscillators and spin Hamiltonians.

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## 1. Introduction

The problem of the solution of the time-dependent Schrödinger equation,

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{1}$$

is as old as quantum mechanics. It is well known that this equation may be reduced to the time-independent Schrödinger equation, i.e. the eigenvalue equation for the Hamiltonian, provided that the eigenstates of the Hamiltonian are time independent[1]. The search for exact solutions of the eigenvalue equation for the Hamiltonian has been an ongoing effort for the past seven decades. A rather recent development in this direction is the application of the ideas of supersymmetric quantum mechanics [2]. The main ingredient provided by supersymmetry is that the eigenvectors of the bosonic and fermionic partner Hamiltonians are related by a supersymmetry transformation [3]. Therefore, one can construct the solutions of the eigenvalue problem for one of the partner Hamiltonians, if the other is exactly solvable. In general, this method cannot be used to relate the solutions of the time-dependent Schrödinger equation unless the partner Hamiltonians have time-independent eigenvectors. The aim of this paper is to explore the utility of supersymmetry in solving time-dependent Schrödinger equation for a general class of time-dependent Hamiltonians.

[1] In this case, the adiabatic approximation is exact [1].

This problem has been considered by Bagrov and Samsonov [4] and Cannata *et al* [5] for the standard Hamiltonians of the form $H = p^2/(2m) + V(x;t)$ in one dimension. Our method differs from those of these authors in the following way. First, we approach the problem from the point of view of the theory of dynamical invariants [6, 7]. Dynamical invariants are certain (time-dependent) operators with a complete set of eigenvectors that are exact solutions of the time-dependent Schrödinger equation. We can easily use the ideas of supersymmetric quantum mechanics to relate the solutions of the time-dependent Schrödinger equation for two different Hamiltonians, if we can identify them with the bosonic and fermionic Hamiltonians of a (not necessarily supersymmetric) $\mathbb{Z}_2$-graded quantum system admitting a supersymmetric dynamical invariant. Unlike [4] and [5], we consider general even supersymmetric dynamical invariants and use our recent results on the geometrically equivalent quantum systems [8] to give a complete characterization of the time-dependent Hamiltonians that admit supersymmetric dynamical invariants.

The organization of the paper is as follows. In section 2, we present a brief review of the dynamical invariants and survey our recent results on identifying the Hamiltonians that admit a given dynamical invariant. In section 3, we discuss the supersymmetric dynamical invariants and we give a characterization of the quantum systems that admit a Hermitian supersymmetric dynamical invariant. In sections 4 and 5, we apply our general results to obtain classes of exactly solvable time-dependent generalized harmonic oscillators and spin systems, respectively. In section 6, we compare our method with that of [4] and [5] and present our concluding remarks.

## 2. Dynamical invariants

By definition [6, 7], a dynamical invariant is a nontrivial solution $I(t)$ of the Liouville–von-Neumann equation

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) = \mathrm{i}[I(t), H(t)] \tag{2}$$

where $H(t)$ denotes the Hamiltonian.

Consider a Hermitian Hamiltonian $H(t)$ admitting a Hermitian dynamical invariant $I(t)$, and suppose that $I(t)$ has a discrete spectrum[2]. Then, equation (2) may be used to show that the eigenvalues $\lambda_n$ of $I(t)$ are constant and the eigenvectors $|\lambda_n, a; t\rangle$ yield the evolution operator $U(t)$ for the Hamiltonian $H(t)$ according to

$$U(t) = \sum_n \sum_{a=1}^{d_n} u^n_{ab}(t) |\lambda_n, a; t\rangle\langle\lambda_n, b; 0|. \tag{3}$$

Here $n$ is a spectral label, $a \in \{1, 2, \ldots, d_n\}$ is a degeneracy label, $d_n$ is the degree of degeneracy of $\lambda_n$, the eigenvectors $|\lambda_n, a; t\rangle$ are assumed to form a complete orthonormal basis of the Hilbert space, $u^n_{ab}(t)$ are the entries of the solution of the matrix Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} u^n(t) = \Delta(t) u^n(t) \qquad u^n(0) = 1 \tag{4}$$

$$\Delta(t) := \mathcal{E}^n(t) - \mathcal{A}^n(t) \tag{5}$$

and $\mathcal{E}^n(t)$ and $\mathcal{A}^n(t)$ are matrices with entries

$$\mathcal{E}^n_{ab} := \langle\lambda_n, a; t|H(t)|\lambda_n, b; t\rangle \qquad \mathcal{A}^n_{ab} := \mathrm{i}\langle\lambda_n, a; t|\frac{\mathrm{d}}{\mathrm{d}t}|\lambda_n, b; t\rangle \tag{6}$$

respectively [7, 9]. Note that $\mathcal{E}^n(t)$, $\mathcal{A}^n(t)$, $\Delta^n(t)$ are Hermitian matrices and $u^n(t)$ is unitary.

[2] The generalization to a continuous spectrum is not difficult.

In view of equation (3),

$$|\psi_n, a; t\rangle := U(t)|\lambda_n, a; 0\rangle = \sum_{b=1}^{d_n} u_{ba}^n(t)|\lambda_n, b; t\rangle \tag{7}$$

are solutions of the Schrödinger equation (1). These solutions actually form a complete orthonormal set of eigenvectors of $I(t)$. We may use this observation or alternatively equation (3) to show

$$I(t) = U(t)I(0)U^\dagger(t). \tag{8}$$

Now, suppose that $I(t)$ is obtained from a parameter-dependent operator $I[\bar{R}]$ as $I(t) = I[\bar{R}(t)]$ where:

(1) $\bar{R} = (\bar{R}^1, \bar{R}^2, \dots, \bar{R}^r)$, $\bar{R}^i$ are real parameters denoting the coordinates of points of a parameter manifold $\bar{M}$;
(2) $\bar{R}(t)$ determines a smooth curve in $\bar{M}$;
(3) $I[\bar{R}]$ is a Hermitian operator with a discrete spectrum;
(4) (in local coordinate patches of $\bar{M}$) the eigenvectors $|\lambda_n, a; \bar{R}\rangle$ of $I[\bar{R}]$, i.e. the solutions of

$$I[\bar{R}]|\lambda_n, a; \bar{R}\rangle = \lambda_n|\lambda_n, a; \bar{R}\rangle \qquad \text{with} \quad a \in \{1, 2, \dots, d_n\} \tag{9}$$

are smooth (single-valued) functions of $\bar{R}$;
(5) $\lambda_n$ and $d_n$ are independent of $\bar{R}$;
(6) $|\lambda_n, a; \bar{R}\rangle$ form a complete orthonormal basis.

In the following, we shall identify $|\lambda_n, a; t\rangle$ with $|\lambda_n, a; \bar{R}(t)\rangle$ and express $|\lambda_n, a; \bar{R}\rangle$ in the form

$$|\lambda_n, a; \bar{R}\rangle = W[\bar{R}]|\lambda_n, a; \bar{R}(0)\rangle \tag{10}$$

where $W[\bar{R}]$ is a unitary operator and $W = W[\bar{R}]$ defines a single-valued function of $\bar{R}$. Equations (9) and (10) suggest

$$I[\bar{R}] = W[\bar{R}]I(0)W[\bar{R}]^\dagger. \tag{11}$$

For a closed curve $\bar{R}(t)$, there exists $T \in \mathbb{R}^+$ such that $\bar{R}(T) = \bar{R}(0)$, and the quantity

$$\Gamma^n(T) := \mathcal{T}e^{i\int_0^T \mathcal{A}^n(t')\,dt'} = \mathcal{P}e^{i\oint A^n} \tag{12}$$

yields the non-Abelian cyclic geometric phase [7,10,11] associated with the solution $|\psi_n; a; t\rangle$. In equation (12), $\mathcal{T}$ and $\mathcal{P}$ respectively denote the time-ordering and path-ordering operators, the loop integral is over the closed path $\bar{R}(t)$ and $A^n$ is the nondegenerate non-Abelian generalization of the Berry connection one-form [12, 13]. The latter is defined in terms of its matrix elements:

$$A_{ab}^n[\bar{R}] := i\langle\lambda_n, a; \bar{R}|\bar{d}|\lambda_n, b; \bar{R}\rangle \tag{13}$$

where $\bar{d} = \sum_i d\bar{R}^i\, \partial/\partial\bar{R}^i$ is the exterior derivative operator on $\bar{M}$. If $\lambda_n$ is nondegenerate, $\Gamma^n(t)$ is just a phase factor. It coincides with the (nonadiabatic) geometric phase of Aharonov and Anandan [14].

Next, we introduce $W(t) := W[\bar{R}(t)]$. Then as discussed in [8], Hermitian Hamiltonians that admit the invariant

$$I(t) = W(t)I(0)W(t)^\dagger \tag{14}$$

have the form

$$H(t) = W(t)Y(t)W(t)^\dagger - iW(t)\frac{d}{dt}\,W(t)^\dagger \tag{15}$$

where $Y(t)$ is any Hermitian operator commuting with $I(0)$. Note that according to equation (15), $H(t)$ is related to $Y(t)$ by a time-dependent (canonical) unitary transformation of the Hilbert space [7, 8, 15, 16], namely $|\psi(t)\rangle \to W(t)|\psi(t)\rangle$. This observation may be used to express the evolution operator $U(t)$ of $H(t)$ in the form

$$U(t) = W(t)V(t) \tag{16}$$

where $V(t) := \mathcal{T}\mathrm{e}^{-\mathrm{i}\int_0^t Y(t')\,\mathrm{d}t'}$ is the evolution operator for $Y(t)$. Note that $Y(t)$ commutes with $I(0)$, therefore if $I(0)$ has a nondegenerate spectrum, $Y(t)$ has a constant eigenbasis. In this case, $Y(t)$ with different $t$ commute and

$$V(t) = \mathrm{e}^{-\mathrm{i}\int_0^t Y(t')\,\mathrm{d}t'}. \tag{17}$$

Having expressed $U(t)$ in terms of $Y(t)$ and $W(t)$, we can write the solutions (7) of the Schrödinger equation in the form

$$|\psi_n, a; t\rangle := W(t)V(t)|\lambda_n, a; 0\rangle = \sum_{b=1}^{d_n} V^n_{ab}(t)W(t)|\lambda_n, b; 0\rangle = \sum_{b=1}^{d_n} V^n_{ab}(t)|\lambda_n, b; t\rangle \tag{18}$$

where $V^n_{ab}(t) := \langle\lambda_n, a; 0|V(t)|\lambda_n, b; 0\rangle$. If $Y(t)$ with different values of $t$ commute, this equation takes the form

$$|\psi_n, a; t\rangle := \mathrm{e}^{-\mathrm{i}\int_0^t y^n_a(t')\,\mathrm{d}t'}|\lambda_n, a; t\rangle$$

where $y^n_a(t) := \langle\lambda_n, a; 0|Y(t)|\lambda_n, a; 0\rangle$.

We conclude this section by emphasizing that equations (15) and (16) are valid for any time-dependent unitary operator $W(t)$ satisfying equation (14). For example, one may identify $W(t)$ with the evolution operator of another Hamiltonian that admits the same invariant $I(t)$. Note that in general such a choice of $W(t)$ cannot be expressed as the image of a curve $\bar{R}(t)$ under a single-valued function $W[\bar{R}]$. In particular, $|\lambda_n, a; t\rangle' := W(t)|\lambda_n; a; 0\rangle$ cannot be written as $|\lambda_n, a; \bar{R}(t)\rangle$ for parameter-dependent vectors $|\lambda_n, a; \bar{R}\rangle$ that are single-valued functions of $\bar{R}$. This in turn implies that $|\lambda_n, a; t\rangle'$ cannot be used in the calculation of the geometric phases.

## 3. $\mathbb{Z}_2$-graded systems admitting supersymmetric invariants

A $\mathbb{Z}_2$-graded quantum system [17] is a system whose Hilbert space $\tilde{\mathcal{H}}$ is the direct sum of two of its nontrivial subspaces $\mathcal{H}_\pm$, i.e. $\tilde{\mathcal{H}} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and whose Hamiltonian maps $\mathcal{H}_\pm$ to $\mathcal{H}_\pm$. The elements of $\mathcal{H}_+$ and $\mathcal{H}_-$ are respectively called bosonic and fermionic state vectors, or graded state vectors with definite grading (or chirality) zero and unity. Operators preserving the grading of the graded state vectors are called even operators. Those that change the grading of these state vectors are called odd operators.

In the two-component representation of the Hilbert space, where the first component $|\psi_+\rangle$ denotes the bosonic and the second component $|\psi_-\rangle$ denotes the fermionic part of a state vector $|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle$, the Hamiltonian has the form

$$H(t) = \begin{pmatrix} H_+(t) & 0 \\ 0 & H_-(t) \end{pmatrix}. \tag{19}$$

Here $H_+(t) : \mathcal{H}_+ \to \mathcal{H}_+$ and $H_-(t) : \mathcal{H}_- \to \mathcal{H}_-$ are Hermitian operators. They are respectively called the bosonic and fermionic Hamiltonians.

Now, suppose that $\mathcal{H}_+ = \mathcal{H}_- =: \mathcal{H}$ and consider a parameter-dependent odd operator $\mathcal{Q} = \mathcal{Q}[\bar{R}]$ and an even Hermitian operator $I = I[\bar{R}]$ that satisfy the algebra of $N = 1$ supersymmetric quantum mechanics [3]:

$$\mathcal{Q}^2 = 0 \qquad [\mathcal{Q}, I] = 0 \qquad \{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2I. \tag{20}$$

In particular, suppose that in the two-component representation of the Hilbert space

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}$$

where $d = d[\bar{R}]$ is a linear operator. This choice of $\mathcal{Q}$ satisfies the superalgebra (20) provided that

$$I = \begin{pmatrix} I_+ & 0 \\ 0 & I_- \end{pmatrix} \tag{21}$$

with

$$I_+ := \tfrac{1}{2}\, d^\dagger d \qquad I_- := \tfrac{1}{2}\, dd^\dagger. \tag{22}$$

As is well known from the study of the spectral properties of supersymmetric systems, one can use equations (22) to derive the following properties of $I_\pm$.

- $I_+$ and $I_-$ have non-negative spectra with the same set of positive eigenvalues $\lambda_n$.
- The degree of degeneracy $d_n$ of $\lambda_n > 0$ as an eigenvalue of $I_+$ is the same as its degree of degeneracy as an eigenvalue of $I_-$.
- Orthonormal eigenvectors $|\lambda_n, a, \pm; \bar{R}\rangle$ of $I_\pm[\bar{R}]$ associated with $\lambda_n > 0$ are related according to

$$d[\bar{R}]|\lambda_n, a, +; \bar{R}\rangle = \sqrt{2\lambda_n}\sum_{b=1}^{d_n} v_{ba}[\bar{R}]|\lambda_n, b, -; \bar{R}\rangle \tag{23}$$

$$d^\dagger[\bar{R}]|\lambda_n, b, -; \bar{R}\rangle = \sqrt{2\lambda_n}\sum_{a=1}^{d_n} v_{ab}[\bar{R}]^\dagger|\lambda_n, a, +; \bar{R}\rangle \tag{24}$$

where $v_{ab}[\bar{R}]$ are the entries of a unitary $d_n \times d_n$ matrix $v[\bar{R}]$. In particular, for a given orthonormal set $\{|\lambda_n, a, +; \bar{R}\rangle \mid \lambda_n > 0\}$ of the eigenvectors of $I_+[\bar{R}]$,

$$|\lambda_n, a, -; \bar{R}\rangle := (2\lambda_n)^{-1/2} d[\bar{R}]|\lambda_n, a, +; \bar{R}\rangle$$

form a complete orthonormal eigenbasis of $I_-[\bar{R}]$ for $\mathcal{H}_- - \mathrm{Ker}(I_-[\bar{R}])$. Here 'Ker' denotes the kernel or the eigenspace with zero eigenvalue.

Next, we introduce $I(t) := I[\bar{R}(t)]$ and $I_\pm(t) := I_\pm[\bar{R}(t)]$ for some curve $\bar{R}(t)$ in the parameter space $\bar{M}$ and demand that $I(t)$ is a dynamical invariant for the Hamiltonian $H(t)$. In view of equations (2), (19), (21), and (22), $I_\pm(t)$ is a dynamical invariant for $H_\pm(t)$.

We can write $I_\pm[\bar{R}]$ in the form (11) by requiring $d[\bar{R}]$ to satisfy

$$d[\bar{R}] = W_-[\bar{R}]d(0)W_+[\bar{R}]^\dagger \tag{25}$$

where $d(t) := d[\bar{R}(t)]$ and $W_\pm[\bar{R}]$ fulfil

$$|\lambda_n, a, \pm; \bar{R}\rangle = W_\pm[\bar{R}]|\lambda_n, a, \pm; \bar{R}(0)\rangle. \tag{26}$$

In view of equations (22) and (25),

$$I_\pm[\bar{R}] = W_\pm[\bar{R}]I_\pm(0)W_\pm[\bar{R}]^\dagger. \tag{27}$$

Moreover, employing equations (15) and (16), we can express the Hamiltonians $H_\pm(t)$ and their evolution operators $U_\pm(t)$ in the form

$$H_\pm(t) = W_\pm(t)Y_\pm(t)W_\pm(t)^\dagger - \mathrm{i}W_\pm(t)\frac{\mathrm{d}}{\mathrm{d}t}W_\pm(t)^\dagger \tag{28}$$

$$U_\pm(t) = W_\pm(t)V_\pm(t) \qquad V_\pm(t) := \mathcal{T}\,\mathrm{e}^{-\mathrm{i}\int_0^t Y_\pm(t')\,\mathrm{d}t'} \tag{29}$$

where $Y_\pm(t)$ are Hermitian operators satisfying

$$[Y_\pm(t), I_\pm(0)] = 0. \tag{30}$$

For example, we can choose $Y_\pm(t) = P_{\rm t}(I_\pm(0))$ where $P_{\rm t}$ is a polynomial with time-dependent coefficients.

In view of equation (18), we have the following set of orthonormal solutions of the Schrödinger equation for the Hamiltonians $H_\pm(t)$.

$$|\psi_n, a, \pm; t\rangle = \sum_{b=1}^{d_n} V^n_{ab\pm}(t)|\lambda_n, b, \pm; t\rangle \tag{31}$$

where $V^n_{ab\pm}(t) := \langle\lambda_n, a, \pm; 0|V_\pm(t)|\lambda_n, b, \pm; 0\rangle$. As we discussed above, given an eigenbasis $|\lambda_n, a, +; t\rangle$ for $I_+(t)$ we can set

$$|\lambda_n, a, -; t\rangle = (2\lambda_n)^{-1/2} d(t)|\lambda_n, a, +; t\rangle \qquad \text{for} \quad \lambda_n > 0.$$

This identification may be used to relate the solutions (31) according to

$$|\psi_n, a, -; t\rangle = (2\lambda_n)^{-1/2} \sum_{b,c=1}^{d_n} V^{n*}_{ca+}(t) V^n_{cb-}(t)\, d(t)|\psi_n, b, +; t\rangle \qquad \forall\lambda_n > 0. \tag{32}$$

For the case where $Y_\pm(t)$ with different $t$ commute, equations (31) and (32) take the form

$$|\psi_n, a, \pm; t\rangle = {\rm e}^{-{\rm i}\int_0^t y^n_{a\pm}(t')\,{\rm d}t'}|\lambda_n, a, \pm; t\rangle \tag{33}$$

$$|\psi_n, a, -; t\rangle = (2\lambda_n)^{-1/2}{\rm e}^{{\rm i}\int_0^t [y^n_{a+}(t') - y^n_{a-}(t')]\,{\rm d}t'}\, d(t)|\psi_n, a, +; t\rangle \qquad \forall\lambda_n > 0 \tag{34}$$

respectively. Here $y^n_{a\pm}(t) := \langle\lambda_n, a, \pm; 0|Y_\pm(t)|\lambda_n, a, \pm; 0\rangle$.

The above construction is valid for any choice of time-dependent unitary operators $W_\pm(t)$ satisfying

$$d(t) = W_-(t) d(0) W_+(t)^\dagger. \tag{35}$$

These observations together with equation (8) suggest a method of generating a class of exactly solvable time-dependent Schrödinger equations. This is done according to the following prescription.

(1) Choose a Hamiltonian $H_+(t)$ whose time-dependent Schrödinger equation is exactly solvable, i.e., its evolution operator $U_+(t)$ is known.
(2) Choose an arbitrary constant operator $d_0$ and a unitary operator $W_-(t)$ satisfying $W_-(0) = 1$.
(3) Set $I_+(0) := d_0^\dagger d_0/2$, $I_-(0) := d_0 d_0^\dagger/2$, and $W_+(t) = U_+(t)$. Then, by construction $I_+(t) := U_+(t) I_+(0) U_+^\dagger$ is a dynamical invariant for $H_+(t)$. It also satisfies $I_+(t) = d(t)^\dagger d(t)/2$ for $d(t) := W_-(t) d_0 U_+(t)^\dagger$. Note that, in view of equation (28) and the Schrödinger equation

$$\mathrm{i}\frac{\rm d}{{\rm d}t} U_+(t) = H_+(t) U_+(t) \tag{36}$$

this choice of $W_+(t)$ corresponds to taking $Y_+(t) = 0$.
(4) Let $Y_-(t)$ be a Hermitian operator commuting with $I_-(0)$. Then according to equation (28), $H_+(t)$ and

$$H_-(t) := W_-(t) Y_-(t) W_-(t)^\dagger - {\rm i} W_-(t) \frac{\rm d}{{\rm d}t} W_-(t)^\dagger \tag{37}$$

are partner Hamiltonians, and $H_-(t)$ admits the invariant $I_-(t) := W_-(t) I_-(0) W_-(t)^\dagger$.

The choice $W_+(t) = U_+(t)$ also implies that $|\lambda_n, a, +; t\rangle = U_+(t)|\lambda_n, a, +; 0\rangle$ are solutions of the Schrödinger equation (1) for $H_+(t)$. Furthermore, for all $\lambda_n > 0$,

$$|\psi_n, a, -; t\rangle = (2\lambda_n)^{-1/2} \sum_{b=1}^{d_n} V^n_{ab-}(t) d(t) |\lambda_n, b, +; t\rangle \tag{38}$$

are solutions of the Schrödinger equation for the Hamiltonian $H_-(t)$. These solutions span $\mathcal{H}_- - \mathrm{Ker}(I_-(0)) = \mathcal{H}_- - \mathrm{Ker}(d_0)$. Again if $Y_-(t)$ with different values of $t$ commute, we have

$$|\psi_n, a, -; t\rangle = (2\lambda_n)^{-1/2} \mathrm{e}^{-\mathrm{i}\int_0^t y^n_{a-}(t')\,\mathrm{d}t'} d(t) |\lambda_n, a, +; t\rangle \qquad \forall \lambda_n > 0. \tag{39}$$

One can also employ an alternative construction for the Hamiltonian $H_-(t)$ in which one still defines $I_+(t)$ according to $I_+(t) := U_+(t) I_+(0) U_+(t)^\dagger$ but uses another unitary operator $W_+(t)$ to express it as $I_+(t) = W_+(t) I_+(0) W_+(t)^\dagger$. In this way, one may choose $W_+(t)$ to be the image of a curve $\bar{R}(t)$ in a parameter space $\bar{M}$ under a single-valued function $W_+ = W_+[\bar{R}]$. This is especially convenient for addressing the geometric phase problem for the Hamiltonians $H_\pm(t)$. Following this approach, one must determine $Y_+(t)$ according to equation (28), i.e.

$$Y_+(t) = W_+(t)^\dagger H_+(t) W_+(t) - \mathrm{i} W_+(t)^\dagger \frac{\mathrm{d}}{\mathrm{d}t} W_+(t). \tag{40}$$

One then obtains the Hamiltonian $H_-(t)$ by substituting (40) in (28).

## 4. Partner Hamiltonians for the unit simple harmonic oscillator Hamiltonian

In this section we explore the partner Hamiltonians for the Hamiltonian of the unit simple harmonic oscillator:

$$H_+ = \tfrac{1}{2}(p^2 + x^2) = a^\dagger a + \tfrac{1}{2} = a\,a^\dagger - \tfrac{1}{2}. \tag{41}$$

Here $p$ and $x$ are respectively the momentum and position operators and $a := (x + \mathrm{i}p)/\sqrt{2}$. Let $W_-(t)$ be a unitary operator satisfying $W_-(0) = 1$ and

$$d(t) := W_-(t) a^\dagger. \tag{42}$$

Then $I_+ = d^\dagger d/2 = a\,a^\dagger/2$ is a dynamical invariant for $H_+$. This invariant together with

$$I_-(t) = \tfrac{1}{2} d(t) d(t)^\dagger = \tfrac{1}{2} W_-(t) a^\dagger a\, W_-(t)^\dagger \tag{43}$$

form a supersymmetric dynamical invariant. The associated 'fermionic' partner Hamiltonian is given by equation (37) where $Y_-(t)$ is a Hermitian operator commuting with $I_-(0) = a^\dagger a/2$.

For example, let

$$Y_-(t) = \frac{f(t)}{4}(2a^\dagger a + 1) = \frac{f(t)}{4}(p^2 + x^2) \tag{44}$$

where $f(t)$ is a real-valued function, and $W_-(t) = W_-[\theta(t), \varphi(t)]$ where

$$W_-[\theta, \varphi] := \mathrm{e}^{-\mathrm{i}\varphi K_3} \mathrm{e}^{-\mathrm{i}\theta K_2} \mathrm{e}^{\mathrm{i}\varphi K_3} \tag{45}$$

$$K_1 := \tfrac{1}{4}(x^2 - p^2) \qquad K_2 := -\tfrac{1}{4}(xp + px) \qquad K_3 := \tfrac{1}{4}(x^2 + p^2) \tag{46}$$

$\theta \in \mathbb{R}$, and $\varphi \in [0, 2\pi)$. Note that $Y_-(t)$ with different values of $t$ commute and the operators $K_i$ are generators of the group $SU(1, 1)$ in its oscillator representation. The parameter space of the operator $W_-$ is the unit hyperboloid:

$$\bar{M} = \{(\bar{R}^1, \bar{R}^2, \bar{R}^3) \in \mathbb{R}^3 \mid -(\bar{R}^1)^2 - (\bar{R}^2)^2 + (\bar{R}^3)^2 = 1\}.$$

We have made the choices (44) and (45) for $Y_-(t)$ and $W_-(t)$ in view of the following considerations.

(1) Up to a trivial addition of a multiple of identity, (44) is the most general expression for a second-order differential operator commuting with $I_-(0)$.
(2) Every element of the (oscillator representation of the) Lie algebra of $SU(1,1)$ may be expressed as $W_-Y_-W_-^\dagger$ with $Y_-$ and $W_-$ given by equations (44) and (45), respectively. In particular, as we show in the following, these choices lead to the most general expression for an invariant $I_-(t)$ and a Hamiltonian $H_-(t)$ belonging to (the oscillator representation of the) Lie algebra of $SU(1,1)$.

In order to compute the Hamiltonian $H_-(t)$, we substitute equations (44) and (45) in (37) and use the $su(1,1) = so(2,1)$ algebra,

$$[K_1, K_2] = -\mathrm{i}K_3 \qquad [K_2, K_3] = \mathrm{i}K_1 \qquad [K_3, K_1] = \mathrm{i}K_2$$

and the Backer–Campbell–Hausdorff formula to compute the right-hand side of the resulting equation. We then find, after a rather lengthy calculation,

$$H_-(t) = \sum_{i=1}^{3} R^i(t)K_i \tag{47}$$

where

$$R^1(t) := \sinh\theta(t)\cos\varphi(t)[2f(t) - \dot\varphi(t)] - \sin\varphi(t)\dot\theta(t) \tag{48}$$

$$R^2(t) := \sinh\theta(t)\sin\varphi(t)[2f(t) - \dot\varphi(t)] + \cos\varphi(t)\dot\theta(t) \tag{49}$$

$$R^3(t) := 2\cosh\theta(t)f(t) + [1 - \cosh\theta(t)]\dot\theta(t) \tag{50}$$

and a dot denotes a time derivative.

As seen from equations (46) and (47), $H_-(t)$ is the Hamiltonian of a time-dependent generalized harmonic oscillator [7] with three free functions $f(t)$, $\theta(t)$ and $\varphi(t)$. According to our general analysis, the corresponding Schrödinger equation is exactly solvable. The evolution operator is given by

$$U_-(t) = \mathrm{e}^{-\mathrm{i}\varphi(t)K_3}\mathrm{e}^{-\mathrm{i}\theta(t)K_2}\mathrm{e}^{\mathrm{i}K_3[\varphi(t)-F(t)]}$$

where $F(t) = \int_0^t f(t')\,\mathrm{d}t'$. Furthermore, we can use the stationary solutions of the Schrödinger equation for the unit simple harmonic oscillator (41) to construct solutions of the Schrödinger equation for $H_-(t)$. The stationary solutions for the Hamiltonian (41) are

$$|\psi_n, +; t\rangle := \mathrm{e}^{-\mathrm{i}tE_n}|n\rangle \tag{51}$$

where $E_n = n + 1/2$, $|n\rangle = (n!)^{-1/2}a^{\dagger n}|0\rangle$ and $|0\rangle$ is the ground state vector for the unit simple harmonic oscillator (41) given by $\langle x|0\rangle = \pi^{-1/4}\mathrm{e}^{-x^2/2}$. In view of equations (34), (42), and (51), we have the following orthonormal solutions of the Schrödinger equation for $H_-(t)$.

$$|\psi_n, -; t\rangle = (n+1)^{-1/2}\mathrm{e}^{-\mathrm{i}tE_n}\mathrm{e}^{-\mathrm{i}F(t)K_3}W_-(t)a^\dagger|n\rangle = \mathrm{e}^{-\mathrm{i}\zeta_n(t)}\mathrm{e}^{-\mathrm{i}[F(t)+\varphi(t)]K_3}\mathrm{e}^{-\mathrm{i}\theta(t)K_2}|n+1\rangle \tag{52}$$

where $\zeta_n(t) := [t - \varphi(t)/2]n + t/2 - 3\varphi(t)/4$. Next, we use the identity [18]

$$\mathrm{e}^{\mathrm{i}\theta K_2}|x\rangle = |\mathrm{e}^{\theta/2}x\rangle$$

and the expression for the propagator of the unit simple harmonic oscillator [19], namely

$$U(x, t; x', 0) := \langle x|U(t)|x'\rangle = (2\pi\mathrm{i}\sin t)^{-1/2}\mathrm{e}^{\mathrm{i}[(x^2+x'^2)\cos t - 2xx']/2\sin t}$$

to compute the solutions (52) in the position representation. This yields

$$\langle x|\psi_n, -; t\rangle = \mathrm{e}^{-\mathrm{i}\zeta_n(t)}\int_{-\infty}^{\infty} U(x, F(t)+\varphi(t); x', 0)\phi_{n+1}(\mathrm{e}^{\theta(t)/2}x')\,\mathrm{d}x'$$

where $\phi_n(x) := \langle x|n\rangle$ are the eigenfunctions of the unit simple harmonic oscillator Hamiltonian (41).

## 5. Partner Hamiltonians for the dipole interaction Hamiltonian of a spinning particle in a constant magnetic field

Consider the dipole interaction Hamiltonian of a spinning particle in a constant magnetic field:

$$H = bJ_3 \tag{53}$$

where $b$ is constant (Larmor frequency), the magnetic field is assumed to be directed along the $z$-direction and $J_3$ denotes the $z$-component of the angular momentum operator $\boldsymbol{J} = (J_1, J_2, J_3)$ of the particle. Let $W_-(t)$ be a unitary operator satisfying $W_-(0) = 1$ and

$$d(t) := W_-(t)J_+ \tag{54}$$

where $J_\pm := J_1 \pm \mathrm{i}J_2 = J_\mp^\dagger$. Then, in view of the identity

$$[J_-J_+, J_3] = 0 \tag{55}$$

the operator

$$I_+ = d^\dagger d/2 = J_-J_+/2 \tag{56}$$

is a dynamical invariant for $H_+$. Equation (55) follows from the $su(2) = so(3)$ algebra,

$$[J_1, J_2] = \mathrm{i}J_3 \qquad [J_2, J_3] = \mathrm{i}J_1 \qquad [J_3, J_1] = \mathrm{i}J_2 \tag{57}$$

satisfied by $J_i$, the fact that $\boldsymbol{J}^2$ is a Casimir operator, i.e. $[\boldsymbol{J}^2, J_i] = 0$, and the relation

$$J_-J_+ = J_1^2 + J_2^2 - J_3 = \boldsymbol{J}^2 - J_3(J_3 + 1). \tag{58}$$

The invariant $I_+$ together with

$$I_-(t) = \tfrac{1}{2}\, d(t)d(t)^\dagger = \tfrac{1}{2} W_-(t)J_+J_-W_-(t)^\dagger \tag{59}$$

form a supersymmetric dynamical invariant. The associated 'fermionic' partner Hamiltonian is given by equation (37) where $Y_-(t)$ is a Hermitian operator commuting with $I_-(0) = J_+J_-/2$.

Next, we note that $[J_+J_-, J_3] = 0$. This suggests that we may choose $Y_-(t)$ as a polynomial in $J_3$ with time-dependent coefficients. For example, we may set

$$Y_-(t) = f(t)J_3 \tag{60}$$

where $f(t)$ is a real-valued function. With this choice of $Y_-$, we can construct a class of partner Hamiltonians $H_-(t)$ for $H_+$ representing the dipole interaction of a spinning particle in a time-dependent magnetic field, provided that we choose $W_-(t) = W_-[\theta(t), \varphi(t)]$ according to [7, 20]

$$W_-[\theta, \varphi] := \mathrm{e}^{-\mathrm{i}\varphi J_3}\mathrm{e}^{-\mathrm{i}\theta J_2}\mathrm{e}^{\mathrm{i}\varphi J_3} \tag{61}$$

where $\theta \in [0, \pi)$ and $\varphi \in [0, 2\pi)$. Note that again $Y_-(t)$ with different values of $t$ commute, the parameter space of the operator $W_-$ is the unit sphere, and $\theta$ and $\varphi$ are respectively the polar and azimuthal angles[3].

The calculation of Hamiltonian $H_-(t)$ for these choices of $Y_-$ and $W_-$ is similar to that of section 4. Substituting equations (60) and (61) in (37) and using the $su(2)$ algebra (57) and the Backer–Campbell–Hausdorff formula, we find

$$H_-(t) = \sum_{i=1}^{3} R^i(t)J_i \tag{62}$$

[3] As described in [7, 20], it turns out that $W_-$ as given by equation (61) fails to be single valued at the south pole ($\theta = \pi$). One can alternatively change the sign of $\theta$ on the right-hand side of (61), in which case $W_-$ becomes single valued for all values of $\theta$ and $\varphi$ except for $\theta = 0$, i.e. the north pole.

where

$$R^1(t) := \sin\theta(t)\cos\varphi(t)[2f(t) - \dot\varphi(t)] - \sin\varphi(t)\dot\theta(t) \tag{63}$$

$$R^2(t) := \sin\theta(t)\sin\varphi(t)[2f(t) - \dot\varphi(t)] + \cos\varphi(t)\dot\theta(t) \tag{64}$$

$$R^3(t) := 2\cos\theta(t)f(t) + [1 - \cos\theta(t)]\dot\theta(t). \tag{65}$$

As seen from these equations, the fermionic partner Hamiltonians (62) to the bosonic Hamiltonian (53) also belong to the Lie algebra $su(2)$; they form a three-parameter family of dipole Hamiltonians describing spinning particles in time-dependent magnetic fields. The solution of the Schrödinger equation for this type of Hamiltonian has been extensively studied in the literature. A rather comprehensive list of references may be found in [7].

In view of equations (29), (60), and (61), the evolution operator for the Hamiltonian (62), for arbitrary choices of functions $f$, $\theta$, and $\varphi$, is given by

$$U_-(t) = \mathrm{e}^{-\mathrm{i}\varphi(t)J_3}\mathrm{e}^{-\mathrm{i}\theta J_2}\mathrm{e}^{\mathrm{i}[\varphi(t)-F(t)]J_3} \tag{66}$$

where $F(t) := \int_0^t f(t')\,\mathrm{d}t'$. Moreover, using the supersymmetric nature of our construction, we may construct a set of orthonormal solutions $|\psi_m, -; t\rangle$ of the Schrödinger equation for this Hamiltonian from those of the constant Hamiltonian (53).

In order to compute these solutions, we first note that $W_+(t) = 1$. Therefore, in view of equation (40), $Y_+(t) = H_+(t) = bJ_3$. Furthermore, because $I_+$ commutes with $J_3$, we may set

$$|\lambda_m, +; t\rangle = |j, m\rangle \tag{67}$$

where $|j, m\rangle$ are the well known orthonormal angular basis vectors satisfying

$$J_3|j, m\rangle = m|j, m\rangle \qquad \boldsymbol{J}^2|j, m\rangle = j(j+1)|j, m\rangle \tag{68}$$

$j \in \{0, \pm 1/2, \pm 1, \pm 3/2, \ldots\}$ labels the total angular momentum (spin) of the particle and $m \in \{-j, -j+1, \ldots, j-1, j\}$ is the magnetic quantum number. Now, in view of equations (56), (58) and (67), the eigenvalues of $I_+$ are given by

$$\lambda_m = \langle j, m|I_+|j, m\rangle = j(j+1) - m(m+1). \tag{69}$$

The solutions of the Schrödinger equation for the Hamiltonian (53) that are associated with this choice of $|\lambda_m, +; t\rangle$ are the stationary solutions

$$|\psi_m, +; t\rangle = \mathrm{e}^{-\mathrm{i}tH_+}|j, m\rangle = \mathrm{e}^{-\mathrm{i}btm}|j, m\rangle. \tag{70}$$

Under the supersymmetry transformation, $|\psi_m, +; t\rangle$, with $m < j$, are mapped to the following solutions of the Schrödinger equation for the Hamiltonian (62).

$$|\psi_m, -; t\rangle = \sqrt{\frac{(j-m)(j+m+1)}{2[j(j+1) - m(m+1)]}}\,\mathrm{e}^{\mathrm{i}[(m+1)\varphi(t)-F(t)]}\mathrm{e}^{-\mathrm{i}\varphi(t)J_3}\mathrm{e}^{-\mathrm{i}\theta(t)J_2}|j, m+1\rangle. \tag{71}$$

Note that here $m \in \{-j, -j+1, \ldots, j-2, j-1\}$ and we have made use of equation (34) and the relations

$$y_+(t) = \langle j, m|Y_+(t)|j, m\rangle = bm \qquad y_-(t) = \langle j, m|Y_-(t)|j, m\rangle = f(t)m$$
$$J_+|j, m\rangle = \sqrt{(j-m)(j+m+1)}\,|j, m+1\rangle.$$

Next, consider the special case of the Hamiltonians (62) obtained by choosing $\theta =$ constant and $\varphi = \omega t$ for some $\omega \in \mathbb{R}^+$, namely

$$\begin{aligned} &H_-(t) = br(t)\big\{f_1(t)[\cos(\omega t)J_1 + \sin(\omega t)J_2] + f_2(t)J_3\big\} \\ &r(t) := b^{-1}\sqrt{4f(t)^2 + [\omega - 4f(t)]\omega\sin^2\theta} \\ &f_1(t) := \sin\theta\,[2f(t) - \omega]/[br(t)] \qquad f_2(t) := 2\cos\theta f(t)/[br(t)]. \end{aligned} \tag{72}$$

These correspond to the dipole Hamiltonians for which the direction of the magnetic field precesses about the $z$-axis and its magnitude is an arbitrary function of time. The case of the constant magnitude is obtained by setting $f$ = constant. This is the well known case of a spin in a precessing magnetic field originally studied in [21]. For a more recent treatment see [7, 20].

In section 4, we restricted ourselves to the study of the quadratic invariants (the invariants that are second-order differential operators). This restriction determined the expression for the operator $Y_-(t)$. A choice of $Y_-(t)$ which includes cubic or higher powers of $p$ would lead to the fermionic partner Hamiltonians that cannot be expressed as second-order differential operators. An analogue of this restriction for the systems considered in this section is the condition that $Y_-(t)$ should belong to the Lie algebra $su(2) = so(3)$. Unlike the case of the harmonic oscillators, a violation of this condition does not lead to any serious problem for the spin systems. For example, if we take

$$Y_-(t) = f(t)J_3 + g(t)J_3^2 \tag{73}$$

but keep the same choice for $W_-(t)$, i.e.$\tilde{(}61)$, we are led to a class of exactly solvable fermionic partner Hamiltonians of the form

$$H'_-(t) = H_-(t) + \tilde{H}_-(t) \tag{74}$$

where $H_-(t)$ is given by equation (62) and $\tilde{H}_-(t)$ is a general quadratic Stark Hamiltonian describing the quadrupole interaction of a spinning particle with the magnetic field [22]. A straightforward calculation yields

$$\begin{aligned}&\tilde{H}_-(t) = g(t)W_-(t)J_3^2W_-(t)^\dagger = g(t)[W_-(t)J_3W_-(t)^\dagger]^2 = g(t)\left[\sum_{i=1}^{3}\tilde{R}^i(t)J_i\right]^2\\ &\tilde{R}^1(t) := \sin\theta(t)\cos\varphi(t) \qquad \tilde{R}^2(t) := \sin\theta(t)\sin\varphi(t) \qquad \tilde{R}^3(t) := \cos\theta(t).\end{aligned} \tag{75}$$

The Hamiltonian (75) belongs to the class of quadrupole Hamiltonians

$$\tilde{H}(t) = \sum_{i,j=1}^{3} Q_{ij}(t)J_iJ_j \tag{76}$$

whose algebraic and geometric structure has been studied in [23, 24]. In particular, up to trivial addition of a multiple of identity, any quadrupole Hamiltonian may be written in the form $\tilde{H}(t) = \sum_{\alpha=0}^{4}\rho^\alpha e_\alpha$ where $\rho^\alpha$ are real parameters and

$$\begin{aligned}&e_0 := J_3^2 - \boldsymbol{J}^2/3 \qquad e_1 := (J_1J_3 + J_3J_1)/\sqrt{3} \qquad e_2 := (J_2J_3 + J_3J_2)/\sqrt{3}\\ &e_3 := (J_1^2 - J_2^2)/\sqrt{3} \qquad e_4 := (J_1J_2 + J_2J_1)/\sqrt{3}.\end{aligned}$$

Furthermore, the commutators $[e_\alpha, e_\beta] =: T_{\alpha,\beta}$ generate the group Spin(5) = Sp(2) that acts on the set of all quadrupole Hamiltonians [23][4].

These observations suggest that one may construct supersymmetric dynamical invariants whose bosonic and fermionic components are linear combinations of the generators $T_{\alpha\beta}$; i.e., they belong to the Lie algebra of Spin(5), i.e. $so(5) = sp(2)$. This in turn implies that they may be obtained from constant elements of $so(5)$ by $SO(5)$ rotations [23].

Next, observe that both $e_0$ and $e_4$ commute with $J_3$. Therefore, in our construction of the partner Hamiltonians for the constant dipole Hamiltonian (53), we may take $Y_-(t) = \xi(t)T_{04}$, where $\xi$ is a real-valued function of time. Now, if we take

$$W_-(t) = e^{\sum_{\alpha,\beta=0}^{4} R^{\alpha\beta}(t)T_{\alpha\beta}}$$

[4] The quadrupole Hamiltonians (76) also arise in the study of the adiabatic evolution of a complex scalar field in a Bianchi type IX background spacetime [25].

for arbitrary functions $R^{\alpha\beta}$, we obtain the most general invariant $I_-(t)$ belonging to $so(5)$. By construction, the corresponding fermionic partner Hamiltonians $H_-(t)$ will constitute a large class of time-dependent exactly solvable Hamiltonians belonging to the Lie algebra $so(5)$. The explicit calculation of $H_-(t)$ requires an appropriate parametrization of the operator $W_-$ in terms of the coordinates of its parameter space.

## 6. Discussions and conclusion

In this paper we have studied supersymmetric dynamical invariants. For a given time-dependent Hamiltonian $H_+(t)$, we have constructed a supersymmetric dynamical invariant $I(t)$ and an associated partner Hamiltonian $H_-(t)$ such that the bosonic part of $I(t)$ is a dynamical invariant for $H_+(t)$ and the fermionic part of $I(t)$ is a dynamical invariant for $H_-(t)$. We have shown how the solutions of the Schrödinger equation for $H_+(t)$ may be used to obtain solutions of the Schrödinger equation for $H_-(t)$.

In order to compare our approach with those of [4, 5], we note that we could construct an even supersymmetric invariant of the form (21) by requiring the supersymmetric charge $\mathcal{Q}$ to be a dynamical invariant. It is not difficult to show that substituting $\mathcal{Q}$ in the Liouville–von-Neumann equation yields the intertwining relation

$$d\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - H_+(t)\right] = \left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - H_-(t)\right]d \qquad (77)$$

for the operator $d$. Note that this relation is only a sufficient condition for $I = \{\mathcal{Q}, \mathcal{Q}^\dagger\}/2$ to be a dynamical invariant. This in turn implies that our method is more general than that of [4, 5]. One way to see this is to substitute equation (25) in (77). Using equations (28) and (30), one can then reduce (77) to

$$d_0 Y_+(t) = Y_-(t) d_0.$$

It is not difficult to construct operators $Y_\pm(t)$ that commute with $I_\pm(0)$ but do not satisfy this equation.

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